MINISTRY OF EDUCATION AND TRAINING

HANOI NATIONAL UNIVERSITY OF EDUCATION

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HOLDER CONTINUITY AND STABILITY OF SOLUTIONS TO THE MONGE-AMPERE EQUATIONS

Major: Analytic Mathematics

Code: 9.46.01.02

SUMMARY OF MATHEMATICS DOCTOR THESIS

Ha Noi - 2018

This thesis was done at: Faculty of Mathematics - Imformations
Ha Noi National University of Education

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The thesis will be defended in Ha Noi National University of Education
In return … hour … day …, 2018

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Introduction

1. Rationale

The Monge-Ampère operator is central object of the pluripotential theory, the research direction is attracting attention of many mathematicians around the world, this direction has been strongly developed and gained many achievements by some mathematicians such as P. Åhag, E. Bedford, Z. Błocki, U. Cegrell, L.H. Chinh, R. Czyż, J. P. Demailly, V. Guedj, L.M. Hai, P.H. Hiep, N.X. Hong, T.V. Khanh, N.V. Khue, S. Kołodziej, B.A. Taylor, Y. Xing, A. Zeriahi, and so on in two recent decades.

One of the important issues in investigating the Monge-Ampère operator is the Dirichlet problem $MA(\Omega, \phi, f)$. From 1976 to 2016, the authors achieved so many important results with respect to this problem in the case that Ω is from smooth bounded strictly pseudoconvex domain in $\mathbb{C}^n$ to bounded pseudoconvex domain with C^2 boundary of plurisubharmonic type m. Thus, the problem $MA(\Omega, \phi, f)$ with respect to non-smooth pseudoconvex domains of plurisubharmonic type m is still open issue.

Next, let a sequence of plurisubharmonic functions $\{u_j\}$, one attends to the convergence in C_p-capacity with $p = \{n-1, n\}$, weak convergence of sequence of the respective Monge-Ampère measures $\{(dd^c u_j)^n\}$ and relationship among them. There are so many research results for this problem. More details, the authors pointed out that under certain conditions, the convergence in C_p-capacity with $p = \{n-1, n\}$ of the function sequence $\{u_j\}$ will guarantee weak convergence of sequence of respective complex Monge-Ampère measures $\{(dd^c u_j)^n\}$ and inverse. However, investigating some sufficient conditions to achieve equivalence between the convergence in C_n-capacity of the sequence $\{u_j\}$ and the convergence of sequence of the respective complex Monge-Ampère operators, also from that to study the stability of solutions of complex Monge-Ampère equations is still open issue.

Continue this research direction, one attends to problem of subextensions of a plurisubharmonic function u to larger domain, especially is the maximal subextension functions. Follow this direction, the authors have attended to problem when existence subextension, maximal subextension of u, have also studied many their properties as the Monge-Ampère measure of subextension functions, the maximal

subextension functions. Therefore, the issue of convergence in C_n-capacity of maximal subextension functions is still open problem.

By the above reasons, we choose this research direction with the thesis' topic as follow: *"Holder continuity and stability of solutions to the Monge-Ampere equations"*.

2. Objectives

From the former achievements of this research direction, purpose of thesis is:

- Investigate the problem Dirichlet with respect to complex Monge-Ampère operator on the non-smooth pseudoconvex domain of plurisubharmonic type m.
- Find out sufficient conditions in order to get equivalence between the convergence in C_n-capacity of the sequence of plurisubharmonic functions and the weak convergence of the sequence of the respective complex Monge-Ampère measures.
- Investigate stability of solutions to the complex Monge-Ampère equations.
- Investigate the convergence in C_n-capacity of the sequence of the maximal subextension functions.
- Continue to investigate to find out new issues.

3. Research subjects

- The plurisubharmonic function, the maximal subextension of plurisubharmonic function.
- The plurisubharmonic function classes introduced by U. Cegrell and developed by many authors.
- The complex Monge-Ampère operator.
- The Dirichlet problem with respect to complex Monge-Ampère operator.
- The complex Monge-Ampère equation and their solutions on the function classes Cegrell.
- Properties of convergence in C_n-capacity of the plurisubharmonic functions and maximal subextensions of plurisubharmonic functions.

4. Methodology

- Applying the traditional methods and techniques was used by so many authors to investigate in Complex Analysis.
- Participate in group seminars, organization department seminar to regularly exchange, discuss, study problems as well as new issues.

5. The scientific and practical significance of the thesis

The pluripotential theory, one of the research directions is attracting attention of so many mathematicians. By their applications in the complex analysis of several variables, the complex geometry, the complex partial differential equation, the complex Dynamics, hyperbolic analysis and so on. The thesis' results contribute to the study of results as well as the techniques in this research direction.

6. Research structure

Beside, Table of contents, Introduction, Overview, Conclusion, List of papers used in the thesis, References. The main content of thesis includes three following chapters:

- *Chapter 1. Hölder continuity of solutions to the complex Monge-Ampère equations*
- *Chapter 2. Stabilization of solutions to the complex Monge-Ampère equations*
- *Chapter 3. Maximal subextension of plurisubharmonic function*

Overview

1. Hölder continuity solutions of the complex Monge-Ampre equation in the non-smooth pseudoconvex domains

Let Ω be an open set in $\mathbb{C}^n$ with $n \geq 1$. An upper semi-continuous function $u : \Omega \to [-\infty, +\infty)$ is called plurisubharmonic on Ω if for every complex line l of $\mathbb{C}^n$, $u|_{l\cap\Omega}$ is a subharmonic function in $l \cap \Omega$. The set of plurisubharmonic functions on Ω is denoted by $\mathcal{PSH}(\Omega)$, $\mathcal{PSH}^{-}(\Omega)$ denote set of negative plurisubharmonic functions on Ω and $\mathcal{MPSH}(\Omega)$ denote set of maximal plurisubharmonic functions on Ω. Then, $(dd^c.)^n$ denote complex Monge-Ampère operator. Where $d = \partial + \overline{\partial}$ and $d^c = i\left(\overline{\partial} - \partial\right)$, hence $dd^c = 2i\partial\overline{\partial}$. In 2004, U. Cegrell introduced a classes of unbounded plurisubharmonic functions on bounded hyperconvex domain on which the complex Monge-Ampère can be defined.

Now, we consider the Dirichlet problem for the complex Monge-Ampère equation:

$$MA(\Omega, \phi, f) : \begin{cases} u \in \mathcal{PSH}\left(\Omega\right) \cap L^{\infty}(\Omega) \\ \left(dd^c u\right)^n = f dV \quad \text{in } \Omega \\ \lim\limits_{z\to\xi} u\left(z\right) = \phi\left(\xi\right), \ \forall \xi \in \partial\Omega. \end{cases}$$

Where, f is the non-negative function on Ω, $f \in L^p(\Omega)$ with $p \geq 1$ and ϕ is continuous and bounded on boundary of Ω. With volume form $dV_n = \frac{1}{n!}\beta^n$, $\beta = dd^c\|z\|^2$ is canonical Kähler form $\mathbb{C}^n$. Denoting $u\left(\Omega, \phi, f\right)$ is the solution of problem $MA\left(\Omega, \phi, f\right)$.

When Ω is the bounded strictly pseudoconvex domain with smooth boundary in $\mathbb{C}^n$, in 1976, E. Bedford and B.A. Taylor pointed out that $MA\left(\Omega, \phi, f\right)$ has the unique solution $u(\Omega, \phi, f) \in C^{0,\alpha}(\overline{\Omega})$ as $\phi \in C^{0,2\alpha}(\partial\Omega)$ and $f^{\frac{1}{n}} \in C^{0,\alpha}\left(\overline{\Omega}\right)$ with $0 < \alpha \leq 1$. Next, in 1982, E. Bedford and B.A. Taylor, continue to point out that the problem $MA\left(\Omega, \phi, f\right)$ alway exists the continuous solution $u\left(\Omega, \phi, f\right)$ on $\overline{\Omega}$ if f continue on $\overline{\Omega}$. In 1985, L. Caffarelli, J.J. Kohn, L. Nirenberg and J. Spruck pointed out that if $0 < f \in C^{\infty}\left(\overline{\Omega}\right)$ and $\phi \in C^{\infty}\left(\partial\Omega\right)$ then $MA\left(\Omega, \phi, f\right)$ has a unique plurisubharmonic solution $u \in C^{\infty}\left(\overline{\Omega}\right)$.

In the case Ω is the non-smooth pseudoconvex domain then the problem become much more complex. In 2004, S. Y. Li especially attended to study the problem on the bounded pseudoconvex domain in $\mathbb{C}^n$ with C^2-smooth boundary. He proved

that if Ω is the bounded pseudoconvex domain of plurisubharmonic type m with C^2-smooth boundary, $\phi \in C^{0,m\alpha}(\partial\Omega)$ with $0 < \alpha \leqslant \frac{2}{m}$ and $f^{\frac{1}{n}} \in C^{0,\alpha}\left(\overline{\Omega}\right)$ then $MA(\Omega, \phi, f)$ merely has the unique solution $u \in C^{0,\alpha}\left(\overline{\Omega}\right)$.

In 2008, V. Guedj, S. Kołodziej and A. Zeriahi investigated the problem on the bounded strictly pseudoconvex domains. They pointed out that if $\phi \in C^{1,1}(\partial\Omega)$ and $f \in L^p(\Omega)$ with $p > 1$ then the unique solution u of the problem $MA(\Omega, \phi, f)$ is also α-Hölder continuity. Recently, L. Baracco, T. V. Khanh, S. Pinton and G. Zampieri generalized the result of V. Guedj, S. Kołodziej and A. Zeriahi for the C^2-smooth bounded pseudoconvex domain of plurisubharmonic type m. In the first issue, the thesis attends to study a general result of the theorem in L. Baracco, T. V. Khanh, S. Pinton and G. Zampieri for the non-smooth pseudoconvex domains of plurisubharmonic type m (be able to unbounded).

2. Stabilization of solutions to the complex Monge-Ampère equations

The concept C_n-capacity of the Borel sets has first been introduced and studied by E. Bedford and B. A. Taylor from 1982. In 1996, Y. Xing proved that the Monge-Ampère operator continue under the convergence in C_n-capacity of the sequence of plurisubharmonic functions. Moreover, he also gave a sufficient condition to guarantee for the weak convergence of the sequence of the respective complex Monge-Ampère measures of a sequence of bounded plurisubharmonic functions. Later, in 2008, Y. Xing gained many important results about the relationship between the convergence in capacity of the sequence of plurisubharmonic functions and the weak convergence of the respective complex Monge-Ampère measures.

In 2010, P.H. Hiep investigated the convergence in C_n-capacity of the functions in $\mathcal{E}(\Omega)$. Recent, in 2012 U. Cegrell proved that if a sequence of plurisubharmonic functions bounded from below by a function in the Cegrell's class $\mathcal{E}(\Omega)$ and converge in C_{n-1}-capacity then the respective Monge-Ampère measures also converge in the weak topology.

Moreover, one knows that the convergence in contribution of plurisubharmonic functions in the general case doesn't infer the convergence of their respective Monge-Ampère measures. Therefore, Finding the sufficient conditions in order to by convergence in certain meaning of a sequence of plurisubharmonic functions that follow the weak convergence of the respective Monge-Ampère measures has so much meaningful. Problem is whether able to study some sufficient conditions to achieve equivalence

between the convergence in C_n-capacity of the sequence $\{u_j\}$ and the convergence of sequence of the respective complex Monge-Ampère operators or not. From that, in our next work, we will use the above results to give the general result for the stability theorem of U. Cegrell and S. Kołodziej in 2006.

3. Maximal subextensions of plurisubharmonic functions

The problem that investigate subextensions of plurisubharmonic functions has attracted attention of some authors so soon. The first result in this direction is the theorem of H. El Mir. He gave a example of plurisubharmonic function on the unit bidisc that its restrictions to any smaller bidisc admit no subextension to the whole space. Later, in 1988, E. Bedford and B. A. Taylor proved that for any bounded domain with smooth boundary in $\mathbb{C}^n$ always exists a smooth plurisubharmonic function without subextension to a larger domain.

Thus, when investigate the problem of subextension of plurisubharmonic functions, the authors always attend to conditions to guarantee existence of subextension function. The first result on subextension in Cegrell's classes is due to Cegrell and Zeriahi. For maximal subextension of plurisubharmonic functions, first results are due to Cegrell, Kołodziej and Zeriahi in 2011. They introduced the notion of maximal subextension of plurisubharmonic functions and studied it in the Cegrell's class $\mathcal{F}(\Omega)$. Recently, N.X. Hong proved a result about Monge-Ampère measures of maximal subextensions of plurisubharmonic functions with given boundary values.

In third issue, the thesis will apply the results of chapter 2 about stability of solutions to complex Monge-Ampère equation to investigate the convergence in capacity of maximal subextensions of plurisubharmonic functions with given boundary values.

Chapter 1

Hölder continuity of solutions to the complex Monge-Ampère equations

In this chapter, we will investigate the problem $MA\,(\Omega, \phi, f)$ on non-smooth pseudoconvex domains of plurisubharmonic type m.

1.1 The solution existence of Dirichlet problem

We recall the concept of the hyperconvex domain in $\mathbb{C}^n$ that will need to use in thesis.

Definition 1.1.1. *A bounded domain $\Omega \subset \mathbb{C}^n$ is called hyperconvex if there exists a bounded plurisubharmonic function ρ such that $\{z \in \Omega : \rho(z) < c\} \Subset \Omega$, for every $c \in (-\infty, 0)$.*

Now, we give the general definition of pseudoconvex domain of plurisubharmonic type m (be able to unbounded).

Definition 1.1.2. *Let $m > 0$ and let Ω be a pseudoconvex domain in $\mathbb{C}^n$. We say that Ω is* of plurisubharmonic type m *if there exists a bounded negative function $\rho \in \mathcal{C}^{\frac{2}{m}}(\overline{\Omega})$ such that $\{\rho < -\varepsilon\} \Subset \Omega$, $\forall \varepsilon > 0$ and $\rho(z) - |z|^2$ is plurisubharmonic in Ω.*

Note that every smooth bounded strictly pseudoconvex domain in $\mathbb{C}^n$ is of plurisubharmonic type 2.

Beside proving existence of solutions to the problem $MA\,(\Omega, \phi, f)$ is to investigate its solution properties, especially is Hölder continuity of $u\,(\Omega, \phi, f)$. We have proposition of characteristic of the function class that has Hölder continuity as follows.

Proposition 1.1.3. *Let S be a subset of $\mathbb{C}^n$ and let $\varphi : S \to \mathbb{R}$. Assume that $\alpha > 0$. Then, the following statements are equivalent.*

(a) φ is α-Hölder continuous and bounded on S, i.e.

$$\sup_{\xi\in S} |\varphi(\xi)| + \sup_{\xi,\zeta\in S,\ \xi\neq\zeta} \frac{|\varphi(\xi)-\varphi(\zeta)|}{|\xi-\zeta|^\alpha} < +\infty.$$

(b) There exist $N, \delta_0 > 0$ such that $|\varphi(\xi)| \le N$, $\forall \xi \in S$ and

$$|\varphi(\xi)-\varphi(\zeta)| \le N\delta^\alpha,\ \forall \delta \in (0,\delta_0),\ \forall \xi,\zeta \in S,\ |\xi-\zeta| \le \delta.$$

The set of all α-Hölder continuous functions on S is denoted by $\mathcal{C}^{0,\alpha}(S)$.

Next, we have the proposition of Hölder continuity of solution to problem $MA(\Omega,\phi,0)$.

Proposition 1.1.4. *Let $m > 0$ and let Ω be a pseudoconvex domain of plurisubharmonic type m. Let ρ be as in Definition 1.1.2 and let $\phi \in \mathcal{C}^\alpha(\partial\Omega)$ with $0 < \alpha \le 1$. Define*

$$M := \sup_{\xi\in\partial\Omega} |\phi(\xi)| + \sup_{\xi,\zeta\in\partial\Omega,\xi\neq\zeta} \frac{|\phi(\xi)-\phi(\zeta)|}{|\xi-\zeta|^\alpha}$$

and

$$u = u(\Omega,\phi,0) := \sup\{\varphi \in PSH(\Omega) : \varphi \le \min(\phi(\xi)-h_\xi, M), \forall \xi \in \partial\Omega\},$$

where

$$h_\xi(z) := -4M\left[-\rho(z)+|z-\xi|^2\right]^{\frac{\alpha}{2}},\ z \in \overline{\Omega}, \xi \in \partial\Omega.$$

Then, u is a bounded solution to $MA(\Omega,\phi,0)$. Moreover, $u \in \mathcal{C}^{\min(\frac{\alpha}{m},\alpha)}(\overline{\Omega})$.

Next, also in the same situation of Proposition 1.1.4 we has proposition of bounded solution existence to $MA(\Omega,\phi,f)$ on pseudoconvex domains of plurisubharmonic type m for the case that f has compact support in Ω.

Proposition 1.1.5. *For every $p > 1$ and for every $0 \le f \in L^p(\Omega)$ with compact support in Ω, there exist a constant $A > 0$ and a bounded solution $u(\Omega,\phi,f)$ to $MA(\Omega,\phi,f)$ such that*

$$u(\Omega,\phi,0) + A\rho \le u(\Omega,\phi,f) \le u(\Omega,\phi,0) \text{ on } \overline{\Omega}.$$

where $u(\Omega,\phi,0)$ defined as in Proposition 1.1.4.

Since Theorem 3 of S. Kołodziej in 1996 and Proposition 1.1.5, one has theorem about solution existence of the problem $MA(\Omega,\phi,f)$.

Theorem 1.1.6. *Let $m > 0$ and let Ω be a pseudoconvex domain of plurisubharmonic type m. Let $\phi \in \mathcal{C}^{0,\alpha}(\partial\Omega)$ be bounded with $0 < \alpha \leq 1$ and let $0 \leq f \in L^p(\Omega)$ with $p > 1$. Assume that either Ω is bounded or the support of f is compact on Ω. Then, there exists a bounded solution to $MA(\Omega, \phi, f)$.*

Remark. Note that the uniqueness of solutions in bounded domains implies from Theorem 3.9 of U. Cegrell in 2008. On unbounded domains, the uniqueness of solutions is still open problem.

1.2 Hölder continuity of Dirichlet problem solution

In order to investigate Hölder continuity of solutions to problem $MA\,(\Omega, \phi, f)$, we will apply the technique of V. Guedj, S. Kołodziej and A. Zeriahi in 2008, so we need the concept of C_n-capacity of the Borel sets introduced and investigated first by two authors E. Bedford and B.A. Taylor from 1982.

Definition 1.2.1. Let $\Omega \subset \mathbb{C}^n$ is an open set. If K is a compact subset of Ω. Then, C_n-capacity of K with respect to Ω defined by

$$C_n\,(K, \Omega) := \sup \left\{ \int\limits_K (dd^c u)^n : \; u \in \mathcal{PSH}\,(\Omega)\,, \; -1 \leqslant u \leqslant 0 \right\}.$$

If E is a subset of Ω then

$$C_n\,(E, \Omega) := \sup \left\{ C_n\,(K, \Omega) : \; K \text{ is compact subset of E} \right\}.$$

Note that if E is a Borel set then

$$C_n\,(E, \Omega) = \sup \left\{ \int\limits_E (dd^c u)^n : \; u \in \mathcal{PSH}\,(\Omega)\,, \; -1 \leqslant u \leqslant 0 \right\}.$$

Before present the next contents, we have the following proposition.

Proposition 1.2.2. *Let $m > 0$ and let Ω be a pseudoconvex domain of plurisubharmonic type m. Let $p > 1$ and let $0 \leq f \in L^p(\Omega)$ with compact support on Ω. Assume that $u \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$ such that $(dd^c u)^n = f dV$ in Ω. Then, for every*

$$0 \leq \gamma < \frac{1}{1 + \frac{np}{p-1}},$$

there exists a positive constant A_γ such that

$$\sup_{\Omega}(v-u) \leq A_\gamma \left(\int_{suppf} |u-v| dV \right)^\gamma,$$

for every $v \in PSH(\Omega)$ with $\{u \leq v - \varepsilon\} \Subset \Omega$, $\forall \varepsilon > 0$.

Next, we investigate the problem $MA\,(\Omega, \phi, f)$ on pseudoconvex domains of plurisubharmonic type m (be able to unbounded) for the case that $0 \leq f \in L^p(\Omega)$ has compact support on Ω.

Theorem 1.2.3. *Let $m > 0$ and let Ω be a pseudoconvex domain of plurisubharmonic type m. Let ρ be as in Definition 1.1.2 and let $\phi \in \mathcal{C}^{0,\alpha}(\partial\Omega)$ be bounded with $0 < \alpha \leq 1$. Let $u(\Omega, \phi, 0)$ be as in Propostion 1.1.4. Then, for every $p > 1$ and for every $0 \leq f \in L^p(\Omega)$ with compact support on Ω, there exist a constant $A > 0$ and a bounded solution $u(\Omega, \phi, f)$ to $MA(\Omega, \phi, f)$ such that*

$$u(\Omega, \phi, 0) + A\rho \leq u(\Omega, \phi, f) \leq u(\Omega, \phi, 0) \text{ on } \overline{\Omega}.$$

Moreover, $u(\Omega, \phi, f) \in \mathcal{C}^\gamma(\overline{\Omega})$ for all

$$0 < \gamma < \min\left(\frac{\alpha}{2m}, \frac{\alpha}{2}, \frac{1}{1 + \frac{np}{p-1}} \right).$$

Now, we give the general result for problem $MA\,(\Omega, \phi, f)$ on pseudoconvex domains of plurisubharmonic type m.

Theorem 1.2.4. *Let $m > 0$ and let Ω be a pseudoconvex domain of plurisubharmonic type m. Let $\phi \in \mathcal{C}^{0,\alpha}(\partial\Omega)$ with $0 < \alpha \leq 1$ and let $0 \leq f \in L^p(\Omega)$ with $p > 1$. Assume that either Ω is bounded or the support of f is compact on Ω. Then, there exists a bounded, γ^2-Hölder continuous solution $u(\Omega, \phi, f)$ to $MA(\Omega, \phi, f)$ for all*

$$0 < \gamma < \min\left(\frac{\alpha}{2m}, \frac{\alpha}{2}, \frac{1}{2m(1 + \frac{np}{p-1})}, \frac{1}{2(1 + \frac{np}{p-1})} \right).$$

Chapter 2

Stabilization of solutions to the complex Monge-Ampère equations

In the first part of chapter, we will investigate relationship between the convergence in C_n-capacity of the sequence of plurisubharmonic functions and weak convergence of sequence of the respective complex Monge-Ampère measures.

2.1 Comparison principle for the functions in classes Cegrell

First, we have the concept of the convergence in the capacity of sequence of plurisubharmonic functions as follow.

Definition 2.1.1. *A sequence $\{u_j\} \subset \mathcal{PSH}(\Omega)$, u_j is called convergence to u in C_n-capacity on the set E as $j \to +\infty$, if*

$$\lim_{j\to+\infty} C_n\left(\{|u_j - u| > \delta\}, \Omega\right) = 0, \ \forall \delta > 0.$$

Now we recall some concepts of important function classes introduced by U. Cegrell.

Definition 2.1.2. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. We say that a bounded, negative plurisubharmonic function φ in $\mathcal{PSH}^-(\Omega)$ belongs to $\mathcal{E}_0(\Omega)$ if $\{\varphi < -\varepsilon\} \Subset \Omega$ for all $\varepsilon > 0$ and $\int\limits_\Omega (dd^c\varphi)^n < +\infty$.*

The class $\mathcal{F}(\Omega)$ denote the family of plurisubharmonic functions φ define on Ω that exists a decrease sequence $\{\varphi_j\} \subset \mathcal{E}_0(\Omega)$ that converges pointwise to φ on Ω as $j \to +\infty$ and

$$\sup_j \int\limits_\Omega (dd^c\varphi_j)^n < +\infty.$$

By $\mathcal{E}(\Omega)$ we denote the set of plurisubharmonic functions φ defined on Ω, such that for each open set $G \Subset \Omega$ there exists a plurisubharmonic function $\psi \in \mathcal{F}(\Omega)$ that $\psi = \varphi$ in G.

Next, the class $\mathcal{N}(\Omega)$ was introduced by U. Cegrell in 2008.

Definition 2.1.3. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$. Let $\{\Omega_j\}$ be a increase sequence of hyperconvex domain that $\Omega_j \Subset \Omega_{j+1} \Subset \Omega$ and $\bigcup_{j=1}^{+\infty} \Omega_j = \Omega$. For every $u \in \mathcal{E}(\Omega)$, one put*

$$u_j := \sup \left\{ \varphi \in \mathcal{PSH}^{-}(\Omega) : \ \varphi \leqslant u \ \ in\ \Omega \backslash \Omega_j \right\}$$

and then,

$$\mathcal{N}(\Omega) := \{ u \in \mathcal{E}(\Omega) : \ u_j \nearrow 0 \ \ a.e\ in\ \Omega \}.$$

It is easy to see that $\mathcal{E}_0(\Omega) \subset \mathcal{F}(\Omega) \subset \mathcal{N}(\Omega) \subset \mathcal{E}(\Omega)$.
Let $\mathcal{K} \in \{\mathcal{F}, \mathcal{N}, \mathcal{E}\}$. Denoting $\mathcal{K}^a(\Omega)$ is subset of $\mathcal{K}(\Omega)$ that Monge-Ampre measure $(dd^c.)^n$ vanish on all pluripolar sets of Ω.
Let $f \in \mathcal{E}(\Omega)$ and $\mathcal{K} \in \{\mathcal{F}^a, \mathcal{N}^a, \mathcal{E}^a, \mathcal{F}, \mathcal{N}, \mathcal{E}\}$. Then, we say that a plurisubharmonic function φ defined on Ω belongs to $\mathcal{K}(\Omega, f)$ if there exists $\psi \in \mathcal{K}(\Omega)$ such that

$$\psi + f \leqslant \varphi \leqslant f \text{ in } \Omega.$$

Now, we will see that $u \in \mathcal{N}^a(\Omega, f)$ then pluripolar part of $(dd^c u)^n$ is always carried by $\{f = -\infty\}$.

Proposition 2.1.4. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. Assume that $f \in \mathcal{E}(\Omega)$ and $u \in \mathcal{N}^a(\Omega, f)$ such that $\int\limits_{\Omega} (-\rho)(dd^c u)^n < +\infty$ for some $\rho \in \mathcal{E}_0(\Omega)$. Then*

$$1_{\{u=-\infty\}}(dd^c u)^n = 1_{\{f=-\infty\}}(dd^c f)^n \ \ in\ \Omega.$$

Proposition 2.1.5. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. Let $f \in \mathcal{E}(\Omega)$ and $u \in \mathcal{N}^a(\Omega, f)$ such that $\int\limits_{\Omega} (-\rho)(dd^c u)^n < +\infty$ for some $\rho \in \mathcal{E}_0(\Omega)$. Assume that $v \in \mathcal{E}(\Omega)$ such that $v \leq f$ and $(dd^c v)^n \geq (dd^c u)^n$ in Ω. Then $v \leq u$ on Ω.*

2.2 Convergence in capacity of plurisubharmonic functions

Before present the main result of this part, we have some following results.

Proposition 2.2.1. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and let $f \in \mathcal{E}(\Omega)$. Assume that $\rho \in \mathcal{E}_0(\Omega)$ and $u \in \mathcal{N}^a(\Omega, f)$ such that $\int_\Omega (-\rho)(dd^c u)^n < +\infty$. Then for every $v \in \mathcal{E}^a(\Omega, f)$ and for every $\varphi \in \mathcal{E}_0(\Omega)$ with $\varphi \geq \rho$, we have*

$$\frac{1}{n!} \int\limits_{\{u<v\}} (v-u)^n (dd^c\varphi)^n + \int\limits_{\{u<v\}} -\varphi (dd^c v)^n \leq \int\limits_{\{u<v\}} -\varphi (dd^c u)^n.$$

Lemma 2.2.2. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and let $\{u_j\} \subset \mathcal{E}^a(\Omega)$ such that $u_j \geq u_1$ for every $j \geq 1$ and $u_j \to u_0$ in C_n-capacity in Ω. Assume that $\{\varphi_j^k\}$, $k = 1, 2$ are sequences of uniformly bounded plurisubharmonic functions in Ω which converges weakly to a plurisubharmonic function φ_0^k in Ω. Then*

$$\varphi_j^1 \varphi_j^2 (dd^c u_j)^n \to \varphi_0^1 \varphi_0^2 (dd^c u_0)^n \text{ weakly as } j \to +\infty.$$

As introduced at beginning of chapter, the main result of this section is.

Theorem 2.2.3. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and let $f \in \mathcal{E}(\Omega)$, $w \in \mathcal{N}^a(\Omega, f)$ such that $\int\limits_\Omega (-\rho)(dd^c w)^n < +\infty$ for some $\rho \in \mathcal{E}_0(\Omega)$. Assume that $\{u_j\} \subset \mathcal{N}^a(\Omega, f)$ such that $u_j \to u_0$ a.e. on Ω as $j \to +\infty$ and $u_j \geq w$ in Ω for all $j \geq 0$. Then, the following statements are equivalent.*

(a) $u_j \to u_0$ in C_n-capacity in Ω;

(b) For every $a > 0$, we have

$$\lim_{j \to +\infty} \int\limits_\Omega \max\left(\frac{u_j}{a}, \rho\right) (dd^c u_j)^n = \int\limits_\Omega \max\left(\frac{u_0}{a}, \rho\right) (dd^c u_0)^n.$$

(c) For every $a > 0$, we have

$$\lim_{j \to +\infty} \int\limits_\Omega \left[\max\left(\frac{v_j}{a}, \rho\right) - \max\left(\frac{u_j}{a}, \rho\right)\right] (dd^c u_j)^n = 0,$$

where $v_j := \left(\sup_{k \geq j} u_k\right)^*$.

2.3 Stability of the solutions to complex Monge-Ampère equations

First, we have proposition about existence of solution to the complex Monge-Ampère equation with non-negative Borel measure in the right side.

Proposition 2.3.1. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and let $f \in \mathcal{E}(\Omega)$, $w \in \mathcal{N}^a(\Omega, f)$ such that $\int\limits_\Omega (-\rho)(dd^c w)^n < +\infty$ for some $\rho \in \mathcal{E}_0(\Omega)$. Then for every nonnegative Borel measures μ in Ω such that*

$$(dd^c f)^n \leq \mu \leq (dd^c w)^n,$$

there exists a unique $u \in \mathcal{N}^a(\Omega, f)$ such that $u \geq w$ and $(dd^c u)^n = \mu$ in Ω.

Now, we will use Theorem 2.2.3 to investigate stability of solutions to complex Monge-Ampère equations. More details, we will prove that sequence of their solutions will converge in C_n-capacity when the sequence of respective measures in the right side weakly convergs.

Theorem 2.3.2. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and let $f \in \mathcal{E}(\Omega)$. Assume that $w \in \mathcal{N}^a(\Omega, f)$ such that $\int\limits_\Omega (-\rho)(dd^c w)^n < +\infty$ for some $\rho \in \mathcal{E}_0(\Omega)$. Then for every sequence of nonnegative Borel measures $\{\mu_j\}$ that converges weakly to a non-negative Borel measure μ_0 in Ω and satisfies*

$$(dd^c f)^n \leq \mu_j \leq (dd^c w)^n \text{ for all } j \geq 0,$$

there exist unique $u_j \in \mathcal{N}^a(\Omega, f)$ such that $u_j \geq w$, $(dd^c u_j)^n = \mu_j$ and $u_j \to u_0$ in C_n-capacity in Ω.

Chapter 3

Maximal subextensions of plurisubharmonic function

In this chapter, we will apply the results of former chapter about stability of solutions to complex Monge-Ampère equations to investigate properties of plurisubharmonic function. Especially is the class of the maximal subextension of plurisubharmonic function with given boundary values.

3.1 Properties of functions in the classes Cegrell

First, we give definition of the maximal subextension of a plurisubharmonic function with given boundary values.

Definition 3.1.1. Let $\Omega \subset \hat{\Omega}$ be bounded hyperconvex domains in $\mathbb{C}^n$ and let $u \in \mathcal{PSH}(\Omega)$. A function $\hat{u} \in \mathcal{PSH}(\hat{\Omega})$ is called a subextension of u if $\hat{u} \leq u$ on Ω.

If $f \in \mathcal{E}(\Omega)$, $g \in \mathcal{E}(\hat{\Omega})$, $u \in \mathcal{F}(\Omega, f)$ such that $f \geq g$ on Ω then the function

$$S_{u,g} := \sup\{\varphi \in \mathcal{PSH}(\hat{\Omega}) : \varphi \leq g \text{ in } \hat{\Omega} \text{ and } \varphi \leq u \text{ on } \Omega\}$$

is said to be *maximal subextension* of u with boundary value g.

Remark. When studying the subextension problem in Cegrell's classes with boundary values, one is usually assumed that the function g in the above definition belongs to the class $\mathcal{E}(\hat{\Omega})$. Then, according to the result of R. Czyż and L. Hed in 2008, the existence of the maximal subextension $S_{u,g}$ is justified.

Proposition 3.1.2. *Let $\Omega \subset \hat{\Omega} \Subset \mathbb{C}^n$ be hyperconvex domains and $f \in \mathcal{E}(\Omega)$, $g \in \mathcal{E}(\hat{\Omega})$ such that $f \geq g$ on Ω. Then, if $u \in \mathcal{F}^a(\Omega, f)$ then $S_{u,g} \in \mathcal{F}^a(\hat{\Omega}, g)$.*

Next, we have below proposition that it will be used to prove the main result of chapter.

Proposition 3.1.3. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and G be an open subset of Ω. Assume that $f, g, w \in \mathcal{E}(\Omega)$ and $\{u_j\}, \{v_j\} \subset \mathcal{E}(\Omega)$ satisfying*

(a) $u_j \geq w$ for all $j \geq 1$;

(b) $u_j \to u$ in capacity in Ω;

(c) $v_j \to v$ a.e. in Ω;

(d) $(dd^c u_j)^n = 0$ on $G \cap \{-\infty < u_j < v_j\} \cap \{f < g\}$.

Then, $(dd^c u)^n = 0$ on $G \cap \{-\infty < u < v\} \cap \{f < g\}$.

Now, we prove a comparison principle in the class $\mathcal{F}^a(\Omega, f)$.

Proposition 3.1.4. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and let $f \in \mathcal{E}(\Omega)$. Assume that $u, v \in \mathcal{F}^a(\Omega, f)$ such that*

(i) $(dd^c u)^n = 0$ on $\{-\infty < u < v\}$;

(ii) $\int\limits_\Omega (dd^c u)^n < +\infty$.

Then, $u \geq v$ in Ω.

3.2 Convergence in capacity of maximal subextension functions

First, we have the following proposition as a consequence of Theorem 2.2.3 for the sufficient conditions to guarantee a sequence of plurisubharmonic functions that converges in C_n-capacity.

Proposition 3.2.1. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and let $f \in \mathcal{E}(\Omega)$, $w \in \mathcal{F}^a(\Omega, f)$ such that*

$$\int\limits_\Omega (dd^c w)^n < +\infty.$$

Assume that $\{u_j\}_{j\geq 1} \subset \mathcal{F}^a(\Omega, f)$ such that $u_j \to u_0$ a.e. on Ω as $j \to \infty$ and $u_j \geq w$ in Ω for all $j \geq 0$. Then the following statements are equivalent.

(a) $u_j \to u_0$ in capacity in Ω;

(b) For every $a > 0$ and for every $\rho \in \mathcal{E}_0(\Omega)$, we have

$$\lim_{j\to\infty} \int_\Omega \max\left(\frac{u_j}{a}, \rho\right)(dd^c u_j)^n = \int_\Omega \max\left(\frac{u_0}{a}, \rho\right)(dd^c u_0)^n.$$

(c) For every $a > 0$ and for every $\rho \in \mathcal{E}_0(\Omega)$, we have

$$\lim_{j\to\infty} \int_\Omega \left[\max\left(\frac{v_j}{a}, \rho\right) - \max\left(\frac{u_j}{a}, \rho\right)\right](dd^c u_j)^n = 0,$$

where $v_j := \left(\sup_{k\geq j} u_k\right)^*$.

Next, we have some estimates for maximal subextensions of plurisubharmonic functions with given boundary values and also their Monge-Ampère measures.

Proposition 3.2.2. *Let $\Omega \subset \hat{\Omega}$ be bounded hyperconvex domains in $\mathbb{C}^n$ and let $f \in \mathcal{E}(\Omega)$, $g \in \mathcal{E}(\hat{\Omega})$ with $f \geq g$ on Ω. Assume that $u \in \mathcal{F}^a(\Omega, f)$ such that $u \geq g$ on $\Omega\backslash K$ for some compact set K of Ω and*

$$\int_\Omega (dd^c u)^n < +\infty.$$

Then, $S := S_{u,g} \in \mathcal{F}^a(\hat{\Omega}, g)$ and $(dd^c S)^n \leq 1_{K\cap\{S=u\}}(dd^c u)^n + 1_{\{S=g\}}(dd^c g)^n$ in $\hat{\Omega}$.

Lemma 3.2.3. *Let $\Omega \subset \hat{\Omega}$ be bounded hyperconvex domains in $\mathbb{C}^n$ and let $\{G_j\}$ be a sequence of bounded hyperconvex domains such that $G_j \Subset G_{j+1} \Subset \Omega$ and $\Omega = \bigcup_{j=1}^{\infty} G_j$. Assume that $u \in \mathcal{F}^a(\Omega)$ and define*

$$u_j := \left(\sup\{\varphi \in PSH^-(\Omega) : \varphi \leq u \text{ on } \Omega\backslash G_j\}\right)^*.$$

Then $S_{u_j,0} \nearrow 0$ a.e. in $\hat{\Omega}$ as $j \nearrow \infty$.

Now, one give result about the convergence in C_n-capacity of maximal subextensions with given boundary values, when the sequence of respective plurisubharmonic

functions converges in C_n-capacity.

Theorem 3.2.4. *Let $\Omega \subset \hat{\Omega}$ be bounded hyperconvex domains in $\mathbb{C}^n$ and let $f \in \mathcal{E}(\Omega)$, $g \in \mathcal{E}(\hat{\Omega})$, $w \in \mathcal{F}^a(\Omega, f)$ be such that $f \geq g$ on Ω and*

$$\int_{\hat{\Omega}} (dd^c g)^n + \int_{\Omega} (dd^c w)^n < +\infty.$$

Assume that $\{u_j\} \subset \mathcal{F}^a(\Omega, f)$ is such that $u_j \geq w$ for all $j \geq 1$ and $u_j \to u$ in capacity in Ω. Then, $S_{u_j,g} \to S_{u,g}$ in capacity in $\hat{\Omega}$.

Conclusion and recommendation

I. Conclusion

In this section, we will recall the main results of thesis as follows.

- Proved the existence of the solution to problem $MA(\Omega, \phi, f)$ in the case that Ω is the pseudoconvex domain of plurisubharmonic type m.
- Proved the Hölder continuity for solutions of problem $MA(\Omega, \phi, f)$ in the case that Ω is the pseudoconvex domain of plurisubharmonic type m.
- Gave the conditions with respect to the sequence of plurisubharmonic functions $\{u_j\}$ to have equivalence between the convergence in C_n-capacity of sequence $\{u_j\}$ and weak convergence of the sequence of the respective Monge-Ampère measures $\{(dd^c u_j)^n\}$.
- Proved stability of the solutions to complex Monge-Ampère equations.
- Proved some properties of the maximal subextensions of plurisubharmonic functions $S_{u,g}$ with given boundary values g.
- Proved the convergence in C_n-capacity of the sequence of maximal subextensions $S_{u_j,g}$ of the sequence of plurisubharmonic functions $\{u_j\}$ with given boundary values g as sequence $\{u_j\}$ converges in C_n-capacity.

II. Recommendation

Since the results of thesis in research process, we suggest some next research directions as follows:

- Investigate problem $MA\,(\Omega, \phi, f)$ in the case that Ω is the pseudoconvex domain of plurisubharmonic type m and f is bounded near boundary of Ω.

- Investigate conditions with respect to the sequence of plurisubharmonic functions $\{u_j\}$ in larger classes to have equivalence between the convergence in C_n-capacity of sequence $\{u_j\}$ and weak convergence of the sequence of the respective Monge-Ampère measures $\{(dd^c u_j)^n\}$.

- Investigate stability of solutions to complex Monge-Ampère equations in larger function classes.

Finally, we really expect to receive valuable comments from the readers about research directions and new issues related to the thesis' topic in order to continue developing this research direction.

List of papers used in the thesis

[1] N.X. Hong, N.V. Trao and T.V. Thuy (2017), "Convergence in capacity of plurisubharmonic functions with given boundary values", *Int. J. Math.*, **28**(3), Article Id:1750018, 14p. DOI:10.1142/S0129167X17500185.

[2] N.X. Hong and T.V. Thuy (2018), "Hölder continuous solutions to the complex Monge-Ampère equations in non-smooth pseudoconvex domains", *Anal. Math. Phys.*, **8**, Issue 3, 465-484.

[3] L.M. Hai, T.V. Thuy and N.X. Hong (2018), "A note on maximal subextensions of plurisubharmonic functions", *Acta Math Vietnam*, **43**, 137-146.

Results of the thesis are reported at:

- Conference of Scientific Research and PhD Training, Faculty of Mathematics, Hanoi National University of Eduacation, 2016;
- Seminar of Department of Functional Theory, Faculty of Mathematics, Hanoi National University of Eduacation, 2017;
- Conference of Scientific Research and PhD Training, Faculty of Mathematics, Hanoi National University of Eduacation, 2017;
- 9th Viet Nam Mathematics Congress at Nha Trang, 2018.